The decay rate of solutions to fraction Navier-Stokes equations*

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Abstract: In this paper we derive the decay rate of solutions to the fraction Navier-Stokes equations, considering the property of the semigroup operator $e^{-t(-\Delta)^{\alpha}}$ and L^2-norms. So establish a new and concise method to get the better decay rate in $H^1(R^3)$, which aviods using the Fourier splitting technique completely and relies on a rough decay estimates of $\|\nabla u(t)\|_{L^2}$.

1. Introduction

Consider the fraction Navier-Stokes equations:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p + (-\Delta)^{\alpha} u = 0 \\ divu = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

Where $u=(u_1,u_2,u_3)$ is the velocity field, $p(t,x)$ is a scalar pressure and $\alpha\in(0,1], \Lambda=(-\Delta)^{1/2}$. The study of the incompressible Navier-Stokes equations has a long story. In [1] and [2], Hopf and Leray proved the famous weak solutions to (1). Actually, there are many previous decay results for Leray-Hopf weak solutions, see [2, 9-12] and references therein. In [3], Wiegner did an important work. He proved a decay rate for a Leray-Hopf weak solution by the so called Fourier splitting method, which was first applied to parabolic conservation laws in [8]. In [4, 6] Zhou and Ning got the decay rate of the soutions to the 2-D dissipative quasi-geostrophic flows. In [5], Zhou got the decay rate of the solutions to the 3D Navier-Stokes equations by avioding using the Fourier splitting techique completely.

The purpose of this paper is also to prove the decay rate of the solutions to (1) by the method. First, suppose that the solution is smooth and exists globally which can be achieved if that initial datum is small in [10], so we get a formal proof. Then, we use the argument of Caffarelli et all [7] to make the solution rigorous. So the decay rate established here is valid at least for smooth solutions with small data and suitable weak solutions by Caffarelli, Kohn, and Nirenberg.

This paper is concerned with the decay rate of the decay rate of the solutions to (1) in $H^1(R^3)$. First we consider the linear equation corresponding to (1) with the same initial data

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha} u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{2}$$

The solution of (2) can be represented by the fundamental solution as

$$\varphi(t) = e^{-t\Lambda^{2\alpha}} u_0 = G_{\alpha} * u_0,$$

where G_{α} is given from the Fourier transform as

$$\mathcal{F}(G_{\alpha}(\xi,t)) = e^{-|\xi|^{2\alpha} t}.$$

It is well-known that the L^2 -norm of a weak solution to (1) decays to zero as time goes to infinity. A natural question is how fast does the solution decay in $H^1(R^3)$ .Our main theorem reads as follows.

Theorem Let $u_0 \in H^1(R^3)$ and $\alpha \in (1/2,1)$. If the solution $\varphi(t)$ to (2) corresponding to u_0 satisfies

$$\|\varphi(t)\|_{L^2} \le C(1+t)^{-\beta}, t \ge 0$$

for some $\beta > 0$. Then there exists a weak solution $u(x,t)$ such that

$$\|u(t)\|_{L^2} \le C(1+t)^{-\beta_0}$$

with $\beta_0 = \min\left\{\beta, \frac{5}{4\alpha}\right\}$.

2. A rough decay estimate of $\|\nabla u(t)\|_{L^2}$

In this section we can derive a rough decay estimate for $\|\nabla u(t)\|_{L^2}$ in R^3 by assuming that the solution is smooth. By Parseval's equality, it is easy to prove that

$$\|\Lambda u\|_{L^2}^2 = \int_{R^3} |\xi|^3 \hat{u}^2(\xi) d\xi = \int_{R^3} |\xi \hat{u}(\xi)|^2 d\xi = \|\nabla u\|_{L^2}$$

Therefore, we only need to get the estimate of $\|\Lambda u(t)\|_{L^2}$. We first get the prior estimate which plays an important role in the following sections.

Proposition 2.1 Let $\alpha \in (\frac{1}{2},1)$ **.** If there exists a solution $u(x,t)$ to (1), then it satisfies $\|\nabla u(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$ in $H^1(R^3)$.

To proof the Proposition 2.1, we give Lemma 2.1 and Lemma 2.2 as follows:

Lemma 2.1 Let $u_0 \in L^1(R^3) \cap H^1(R^3)$. Then there exists the solution $u(x,t)$ such that

$$|\hat{u}(\xi,t)| \le \|u_0\|_{L^1} + |\xi| \int_0^t \|u(s)\|_{L^2}^2 \, \mathrm{ds}. \tag{3}$$

Proof. Assume that the solution is smooth in what follows in this section. From (1) we get

$$\partial_t \hat{u} + |\xi|^{2\alpha} \hat{u} = -\mathcal{F}(u \cdot \nabla u).$$

Because of $\nabla \cdot u = 0$, so we have $|\mathcal{F}\{u \cdot \nabla u\}| \le |\xi| \|u(s)\|_{L^2}^2$. Then integrating the above equation with respect to time $[0,t]$,

$$\begin{aligned} |\hat{u}(\xi,t)| &\le |\hat{u}_0(\xi,t)| + |\xi| \int_0^t \|u(s)\|_{L^2}^2 \, ds \\ &\le \|u_0\|_{L^1} + |\xi| \int_0^t \|u(s)\|_{L^2}^2 \, ds \end{aligned} \tag{4}$$

Lemma 2.2 Let $u_0 \in L^1(R^3) \cap H^1(R^3)$ and $\alpha \in [0,1]$. There exists a weak solution u(t) to (1) satisfying $\|u(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2\alpha}}$, where C is the positive real number which depends on $\|u_0\|_{L^1}$ and $\|u_0\|_{L^2}$.

Proof. Multiplying u on both sides of the equation (1) and integrating in R^3 , we have

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\|\Lambda^\alpha u\|_{L^2}^2 = 0 \tag{5}$$

By Plancherel theorem, it follows that

$$\frac{d}{dt}\|\hat{u}(t)\|_{L^2}^2 + 2\left\||\xi|^\alpha \hat{u}\right\|_{L^2}^2 = 0 \tag{6}$$

Now we use Fourier splitting method. Let $B(t) = \{\xi \in R^3 : |\xi| \le g(t)\}$ and R depends on $g(t)$, where $g \in C([0,+\infty]; R^+)$ is undetermined. We get

$$\int_{R^3} |\xi|^{2\alpha} |\hat{u}|^2 d\xi \ge g^{2\alpha} \int_{B(t)^c} |\hat{u}|^2 d\xi$$

$$= g^{2\alpha} \int_{R^3} |\hat{u}|^2 d\xi - g^{2\alpha} \int_{B(t)} |\hat{u}|^2 d\xi \tag{7}$$

By using (7) and Lemma 2.1, we obtain

$$\frac{d}{dt}\int_{R^3} |\hat{u}|^2 d\xi + 2g^{2\alpha} \int_{R^3} |\hat{u}|^2 d\xi \le$$

$$4\pi g^{2\alpha+3}(t)\|u_0\|_{L^1}^2 + 2\pi t g^{2\alpha+5}(t)\int_0^t \|u(s)\|_{L^2}^4 ds \tag{8}$$

Then integrating (8), we get

$$\exp\left(2\int_0^t g^{2\alpha}(s)ds\right)\int_{R^3} |\hat{u}| d\xi \le \|u_0\|_{L^2}^2 + \int_0^t A(s)B(s)ds \tag{9}$$

in $A(s) = \exp(2\int_0^s g^{2\alpha}(\tau)d\tau)$,

$$B(s) = 4\pi g^{2\alpha+3}(s)\|u_0\|_{L^1}^2 + 2\pi s g^{2\alpha+5}(s)\int_0^s \|u(\tau)\|_{L^2}^4 d\tau .$$

To prove proposition 2.1, we first let $g^{2\alpha}(t) = \left(\frac{1}{2} + \frac{1}{2\alpha}\right)[(1+t)\ln(1+t)]^{-1}$

and integrate it on both sides. We have

$$\exp(2\int_0^t g^{2\alpha}(s)ds) = [\ln(1+t)]^{1+\frac{1}{\alpha}}.$$

Then by (9) we can have

$$[\ln(1+t)]^{1+\frac{1}{\alpha}}\|u\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + C\int_0^t \frac{1}{(1+s)^{3/2}} ds.$$

So we get

$$\|u\|_{L^2}^2 \le C[\ln(1+t)]^{-1-\frac{1}{\alpha}} \tag{10}$$

Let $g^{2\alpha}(t) = \frac{1}{2\alpha(1+t)}$ again and integrate it on both sides, and we have

$$\exp\left(2\int_0^t g^{2\alpha}(s)ds\right) = (1+t)^{1+\frac{1}{\alpha}}.$$

Then by (10), we can have

$$\|u\|_{L^2}^2 \le C(1+t)^{-\frac{1}{\alpha}} + C(1+t)^{1-\frac{5}{2\alpha}} \int_0^t \|u\|_{L^2}^2 [\ln(1+s)]^{-1-\frac{1}{\alpha}} ds \quad (11)$$

where C is a positive real number which depends on $\|u_0\|_{L^1}$ and $\|u_0\|_{L^2}$. Now we prove proposition 2.1 by two parts as follows.

Part I $\frac{5}{6} < \alpha < 1.$

If there exists the smooth solution $u(x,t)$ to (1) multi-plying $\Lambda^2 u$ on both sides and integrating in R^3, then

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} |\Lambda u|^2 dx + \int_{R^3} |\Lambda^{\alpha+1} u|^2 dx \le |\int_{R^3} (u\cdot\nabla u)\Lambda^2 u dx| \quad (12)$$

Due to the divergence free of the velocity field u, we have

$$|\int_{R^3} (u\cdot\nabla u)\Lambda^2 u dx| = |\int_{R^3} \mathrm{div}(u\cdot u)\Lambda^2 u dx| \le$$

$$\frac{1}{2}\|\Lambda^{\alpha+1}u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{2-\alpha}(u\cdot u)\|_{L^2}^2 \quad (13)$$

Taking $p = \frac{6}{4\alpha-2}$ and $q = \frac{6}{5-4\alpha}$, then by the embedding lemma in [13] and the fractional type Gagliardo-Nirenberg inequality, we get

$$\|\Lambda^{2-\alpha}(u\cdot u)\|_{L^2}^2 \le C(\|u\|_{L^p}\|\Lambda^{2-\alpha}u\|_{L^q}),$$

$$\|\Lambda^{2-\alpha}u\|_{L^{\frac{6}{5-4\alpha}}}^2 \le C\|\Lambda^{\alpha+1}u\|_{L^2}^2,$$

$$\|u\|_{L^{\frac{6}{4\alpha-2}}}^2 \le C\|u\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\|\Lambda^{\alpha}u\|_{L^2}^{\frac{5-4\alpha}{\alpha}}.$$

Hence (12) can be rewirtten in the following

$$\frac{d}{dt}\|\Lambda u\|_{L^2}^2 \le (C\|u\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\|\Lambda^{\alpha}u\|_{L^2}^{\frac{5-4\alpha}{\alpha}} - 1)\|\Lambda^{\alpha+1}u\|_{L^2}^2. \quad (14)$$

Similary, we can get a differential inequality for $\|\Lambda^{\alpha+1}u\|_{L^2}^2$,

$$\frac{d}{dt}\|\Lambda^{\alpha} u\|_{L^2}^2 \le (C\|u\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\|\Lambda^{\alpha}u\|_{L^2}^{\frac{5-4\alpha}{\alpha}} - 1)\|\Lambda^{\alpha}u\|_{L^2}^2. \quad (15)$$

On the other hand, multiplying equation (1) by u, and integrating for both space and time, then $\|u(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^{\alpha}u(\cdot,s)\|_{L^2}^2 ds = \|u_0\|_{L^2}^2$, for all t ≥ 0. Therefore, there exists a time t_0 such that $\|u(t_0)\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\|\Lambda^{\alpha}u(t_0)\|_{L^2}^{\frac{5-4\alpha}{\alpha}} \le \|u_0\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\|\Lambda^{\alpha}u(t_0)\|_{L^2}^{\frac{5-4\alpha}{\alpha}} \le \frac{1}{C}$,

where C is the bigger constant of these in (14) and (15).

From equation (14) and (15) and the choice of t_0, we have

$$\frac{d}{dt}\|\Lambda u\|_{L^2} \le 0 \text{ and } \frac{d}{dt}\|\Lambda^{\alpha}u\|_{L^2} \le 0,$$

for all $t \ge t_0$. Then we want to obtain the integrability for $\|\Lambda u\|_{L^2}$. Multiplying (1) by $\Lambda^{2-2\alpha}u$, a similar compu-tation yields

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{1-\alpha}u\right\|_{L^2}^2+\left\|\Lambda u\right\|_{L^2}^2\le$$

$$\frac{1}{2}\left\|\Lambda u\right\|_{L^2}^2+C\left\|u\right\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\left\|\Lambda^{\alpha}u\right\|_{L^2}^{\frac{5-4\alpha}{\alpha}}\left\|\Lambda u\right\|_{L^2}^2. \tag{16}$$

Now let $t_1>t_0$ to be such that

$\left\|u(t_1)\right\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\left\|\Lambda^{\alpha}u(t_1)\right\|_{L^2}^{\frac{5-4\alpha}{\alpha}}\le\left\|u_0\right\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\left\|\Lambda^{\alpha}u(t_1)\right\|_{L^2}^{\frac{5-4\alpha}{\alpha}}\le\frac{1}{4C}$. Then integrating (16) with respect to time on $[t_1,t]$,

$\left\|\Lambda^{1-\alpha}u(t)\right\|_{L^2}^2+\int_{t_1}^{t}\left\|\Lambda u(s)\right\|_{L^2}^2ds\le\left\|\Lambda^{1-\alpha}u(t_1)\right\|_{L^2}^2+\quad 2C\int_{t_1}^{t}\left\|u(s)\right\|_{L^2}^{\frac{6\alpha-5}{\alpha}}\left\|\Lambda^{\alpha}u(s)\right\|_{L^2}^{\frac{5-4\alpha}{\alpha}}\left\|\Lambda u(s)\right\|_{L^2}^2ds.$ From $\frac{d}{dt}\left\|\Lambda u\right\|_{L^2}\le 0$ and by the choice of t_1, we obtain $(t-t_1)\left\|\Lambda u(t)\right\|_{L^2}^2\le\int_{t_1}^{t}\left\|\Lambda u(s)\right\|_{L^2}^2ds\le 2\left\|\Lambda^{1-\alpha}u(t_1)\right\|_{L^2}^2$.

So we get a rough decay estimate as

$$\left\|\Lambda u\right\|_{L^2}^2=\left\|\nabla u\right\|_{L^2}^2\le C(1+t)^{-\frac{1}{2}}.$$

Part II $\frac{1}{2}<\alpha\le\frac{5}{6}$.

Multiplying equation (1) by $\Lambda^2 u$ and integrating in R^3, then

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda u\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2\le\frac{1}{4}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+C\left\|\Lambda^{\frac{3}{2}}u\right\|_{L^2}^2. \tag{17}$$

Let $B(t)=\{\xi\in R^3:|\xi|\le g(t)\le R\}$, where R is undetermined.

$$\left\|\Lambda^{\frac{3}{2}}u\right\|_{L^2}^2=\int_{B(t)}|\xi|^3|\hat{u}|^2\,d\xi+\int_{B^c(t)}|\xi|^3|\hat{u}|^2\,d\xi$$

$$\le R^3\left\|u\right\|_{L^2}^2+CR^{1-2\alpha}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2, \tag{18}$$

$$\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2\ge\int_{B^c(t)}|\xi|^{2+2\alpha}|\hat{u}|^2\,d\xi$$

$$\ge R^{2\alpha}\left\|\Lambda u\right\|_{L^2}^2-R^{2+2\alpha}\left\|u\right\|_{L^2}^2. \tag{19}$$

Putting (18) and (19) into (17) and letting R be large enough such that $CR^{1-2\alpha}\le\frac{1}{4}$, then we obtain

$$\frac{d}{dt}\left\|\Lambda u\right\|_{L^2}^2+R^{2\alpha}\left\|\Lambda u\right\|_{L^2}^2\le R^{2+2\alpha}(1+t)^{-\frac{1}{\alpha}}, \tag{20}$$

by using Lemma 2.2.

Then integrating with respect to t, we get

$$\left\|\Lambda u\right\|_{L^2}^2\le\exp(-R^{2\alpha}t)\left\|\Lambda u_0\right\|_{L^2}^2+CR^{2+2\alpha}(1+t)^{-\frac{1}{\alpha}}. \tag{21}$$

Therefore we get $\left\|\Lambda u\right\|_{L^2}^2\le C(1+t)^{-\frac{1}{\alpha}}$.

3. Proof of the main theorem

In this section, we assume the solution is smooth so that the second part of the proof is formal. Then a new, concise and direct method is used to prove theorem in the fundamental the above rough decay estimate.

We present the solution $u(t)$ by the fundamental solution of (1) as

$$u(t) = e^{-\Lambda^{2\alpha}t}u_0 - \int_0^t e^{-\Lambda^{2\alpha}(t-\tau)}(u\cdot\nabla u)ds \tag{22}$$

$$u(t) = e^{-\Lambda^{2\alpha}t}u_0 - \int_0^t \nabla e^{-\Lambda^{2\alpha}(t-\tau)}(u\otimes u)ds \tag{23}$$

So we have

$$\|u(t)\|_{L^2}^2 = \left\|e^{-\Lambda^{2\alpha}t}u_0\right\|_{L^2} + \int_0^t \left\|e^{-\Lambda^{2\alpha}(t-\tau)}(u\cdot\nabla u)\right\|_{L^2} ds \tag{24}$$

$$\|u(t)\|_{L^2}^2 = \left\|e^{-\Lambda^{2\alpha}t}u_0\right\|_{L^2} + \int_0^t \left\|\nabla e^{-\Lambda^{2\alpha}(t-\tau)}(u\otimes u)\right\|_{L^2} ds \tag{25}$$

By Parseval's equality, it follows that

$$\left\|e^{-\Lambda^{2\alpha}(t-\tau)}(u\cdot\nabla u)\right\|_{L^2}^2 = \int_{R^3} e^{-2|\xi|^{2\alpha}t} \,|\,\mathcal{F}(u\cdot\nabla u)\,|^2\, d\xi$$

$$\le \|Du\|_{L^2}^2 t^{-\frac{3}{2\alpha}}. \tag{26}$$

$$\left\|\nabla e^{-\Lambda^{2\alpha}t}(u\otimes u)\right\|_{L^2}^2 \le C\left\|\mathcal{F}(u\otimes u)\right\|_{L^2}^2 \int_0^\infty e^{-2r^{2\alpha}t} r^4 dr$$

$$\le C\|u\|_{L^2}^4 t^{-\frac{5}{4\alpha}}. \tag{27}$$

Combining Proposition 2.1 and (26) together, it follows that

$$\|u(t)\|_{L^2} \le C(1+t)^{-\beta} + C\int_0^t (t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}}\|u(s)\|_{L^2} ds. \tag{28}$$

$$(1+t)^{\beta}\|u(t)\|_{L^2} \le C + C(1+t)^{\beta}Q(t)\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds + C(1+t)^{\beta}Q(t)\int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds \tag{29}$$

By direct computation, we find that

$$\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds = t^{-\frac{3}{4\alpha}}\begin{cases} 1,\text{if}\ \frac{1}{2}+\beta>1 \\ \ln(e+t),\text{if}\ \frac{1}{2}+\beta=1 \\ (1+t)^{1/2-\beta},\text{if}\ \frac{1}{2}+\beta<1. \end{cases}$$

and $\int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds \le t^{1-\frac{3}{4\alpha}}(1+t)^{-\frac{1}{2}-\beta}$.

If $\beta < \dfrac{3}{4\alpha}$, it is true that

$$C(1+t)^{\beta}Q(t)\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds \to 0,\ \ t\to\infty..$$

So there exists a t_0 sufficiently large such that $C(1+t)^{\beta}Q(t)\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}ds \le \frac{1}{2}$, for any $t \ge t_0$.

Then from (29), we have

$$(1+t)^{\beta}\|u(t)\|_{L^2} \le C + \frac{1}{2}Q(t), \text{for } \ t \ge t_0. \tag{30}$$

Let $Q(t) = \max_{t_0 \le s \le t}\{(1+s)^{\beta}\|u(s)\|_{L^2}\}$, then (30) can be reduced to

$$(1+t)^{\beta}Q(t)\|u(t)\|_{L^2} \le C + \frac{1}{2}Q(t_0) + \frac{1}{2}\hat{Q}(t), \tag{31}$$

Now taking the maximum for $t \in [t_0, T]$ on both sides of (31), we obtain

$$\hat{Q}(T) \le C + \frac{1}{2}Q(t_0) + \frac{1}{2}\hat{Q}(T).$$

Therefore,

$$(1+t)^{\beta}\|u(t)\|_{L^2} \le 2C + \max_{0\le s\le t_0}\left\{(1+s)^{\beta}\|u(s)\|_{L^2}\right\} < \infty.$$

Now we consider the case $\beta \ge \frac{3}{4\alpha}$. Since $(1+t)^{-\beta} < (1+t)^{-\frac{3}{4\alpha}}$, it follows from the above step that $\|u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{5\alpha}}$. Thanks to (27) , we obtain that

$$\|u(t)\|_{L^2} \le C(1+t)^{-\beta} + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{5}{4\alpha}}\|u(s)\|_{L^2}^2 ds$$

$$+ C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta}\|u(s)\|_{L^2} ds. \tag{32}$$

Then by the rough estimate, it follows that

$$C\int_0^{\frac{t}{2}}(t-s)^{-\frac{5}{4\alpha}}\|u(s)\|_{L^2}^2 ds \le Ct^{-\frac{5}{4\alpha}}.$$

Taking $\beta_0 = \min\left\{\beta, \frac{5}{4\alpha}\right\}$, (32) is reduced to

$$\|u(t)\|_{L^2} \le C(1+t)^{-\beta_0} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}}\|u(s)\|_{L^2} ds. \tag{33}$$

Multiplying $(1+t)^{\beta_0}$ on both sides of (33) and taking $Q(t) = \max_{0\le s\le t}\{(1+s)^{\beta_0}\|u(s)\|_{L^2}\}$, we have

$$(1+t)^{\beta_0}\|u(t)\|_{L^2} \le C + C(1+t)^{\beta_0}Q(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4\alpha}}(1+s)^{-\frac{1}{2}-\beta_0} ds.$$

$$\le C + CQ(t)(1+t)^{-\frac{1}{2}}t^{1-\frac{3}{4\alpha}}. \tag{34}$$

Similarly, we obtain

$$(1+t)^{\beta_0}\|u(t)\|_{L^2} < \infty.$$

Above all, we complete the theorem of the proof.

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